Some Basics of Classical Logic

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Classical Logic

Propositional and first order predicate logic.

A way to think about a logic (and no more than that) is as consisting of:

- Expressive part: Syntax & Semantics
- Normative part: Logical Consequence (Validity)
    - Derivability (Syntactic Validity)
    - Entailment (Semantic Validity)

Propositional Logic

The language of classical propositional logic consists of:

- Propositional variables: $p, q, r \ldots$
- Logical operators: $\neg$, $\wedge$, $\vee$, $\rightarrow$
- Parentheses: (,)

Propositional Logic

Recursive definition of a well-formed formula (wff) in classical propositional logic:

Definition

1. A propositional variable is a wff.
2. If ϕ and ψ are wffs, then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$ are wffs.
3. Nothing else is a wff.

The recursive nature of the above definition is important in that it allows inductive proofs on the entire logical language.
Exercise: Prove that all wffs have an even number of brackets.

Predicate Logic

The language of classical predicate logic consists of:

- Terms: a) constants $(a, b, c...)$, b) variables $(x, y, z...)$[1]
- n-place predicates $P, Q, R...$
- Logical operators: a) propositional connectives, b) quantifiers $\forall$, $\exists$
- Parentheses: (,)

[1] Functions are also typically included in first order theories.

Predicate Logic

Recursive definition of a wff in classical first-order logic:

Definition

1. If R is a n-place predicate and $t_1, ...t_n$ are terms, then $R(t_1...t_n)$ is a wff.
2. If ϕ and ψ are wffs, and t is a variable, then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $\forall t\phi$ and $\exists t\phi$ are wffs.
3. Only strings of symbols constructed by the previous clauses are wffs.

A special binary predicate '$=$', is often added to the language of predicate logic.

Natural Deduction

- An argument $< \Gamma, \phi >$ consists of a set of premises and a conclusion.
- $\Gamma \vdash \phi$ means that ϕ is derivable from Γ.
- In the special case that the set of premises is the empty set and the conclusion is derivable, we call the conclusion a *theorem*.
- Proofs are finite objects:
  $\Gamma \vdash \phi$ iff there exists finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \phi$ in a finite number of steps.
- The proof system of Natural Deduction consists of rules that allow for the *Introduction* and *Elimination* of the logical operators.
- Some of the rules of ND make use of assumptions which then need to be discharged.

Natural Deduction

Conjunction

$$
\begin{array}{l|ll}
n & \phi & \\
\vdots & \vdots & \\
m & \psi & \\
\vdots & \vdots & \\
l & \phi \land \psi & \land\mathrm{I},\ n,\ m
\end{array}
\qquad
\begin{array}{l|ll}
n & \phi \land \psi & \\
\vdots & \vdots & \\
m & \phi & \land\mathrm{E},\ n
\end{array}
$$

Natural Deduction

Disjunction

$$
\begin{array}{l|ll}
n & \phi & \\
\vdots & \vdots & \\
m & \phi \vee \psi & \vee\mathrm{I},\ n
\end{array}
$$

$$
\begin{array}{l|lll}
n & \phi \vee \psi & & \\
\vdots & \vdots & & \\
m & & \phi & \\
\vdots & & \vdots & \\
s & & \chi & \\
t & & \psi & \\
\vdots & & \vdots & \\
l & & \chi & \\
r & \chi & & \vee\mathrm{E},\ n,\ m\text{–}s,\ t\text{–}l
\end{array}
$$

Natural Deduction

Implication

n	ϕ	
$\vdots$	$\vdots$	
m	ψ	
s	$\phi \to \psi$	$\Rightarrow$I, n, m

n	ϕ	
$\vdots$	$\vdots$	
m	$\phi \to \psi$	
$\vdots$	$\vdots$	
s	ψ	$\Rightarrow$E, n, m

Natural Deduction

Negation[2]

n	ϕ	
$\vdots$	$\vdots$	
m	$\bot$	
$\vdots$	$\vdots$	
s	$\neg\phi$	$\neg$I, n, m

n	$\neg\neg\phi$	
$\vdots$	$\vdots$	
m	ϕ	$\neg\neg$E, n

[2] $\bot$ is used as short for $\phi \wedge \neg\phi$.

Natural Deduction

We can now derive the *law of excluded middle* (LEM): $\phi \vee \neg\phi$.
LEM is a characteristic law of classical logic.
Exercise: Perform the derivation.

The Law of Excluded Middle

1	$\neg(\phi \lor \neg\phi)$	
2	ϕ	
3	$\phi \lor \neg\phi$	$I_\lor$ 2
4	$\bot$	$I_\land$ 1, 3
5	$\neg\phi$	$I_\neg$ 2, 4
6	$\phi \lor \neg\phi$	$I_\lor$ 5
7	$\bot$	$I_\land$ 1, 6
8	$\neg\neg(\phi \lor \neg\phi)$	$I_\neg$ 1, 7
9	$\phi \lor \neg\phi$	DNE 8

Natural Deduction and Intuitionistic Logic

- The rule of *double negation elimination* (DNE) is not an intuitionistically acceptable rule.
- Since DNE is used essentially in the derivation of LEM, this law is not intuitionistically derivable.
- The exclusion of DNE reflects the intuitionistic philosophical idea that there are no mind independent mathematical facts.
- Note that the converse of DNE, that is, $\phi \vdash \neg\neg\phi$, is both classically and intuitionistically derivable.
  Exercise: Perform the derivation.

Natural Deduction and Intuitionistic Logic

Example of a mathematical proof that is not intuitionistically acceptable:

There exist irrational numbers a, b such that a^b is rational.

Proof

- Consider $\sqrt{2}$, which we know to be irrational.
- $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational. (By LEM)
  1. If $\sqrt{2}^{\sqrt{2}}$ is rational: take $a = b = \sqrt{2}$. Then a^b is rational.
  2. If $\sqrt{2}^{\sqrt{2}}$ is irrational: take $a = \sqrt{2}^{\sqrt{2}}$, and $b = \sqrt{2}$.
     Then, $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$, which is rational. $\square$

Natural Deduction and Intuitionistic Logic

The derivable (in the given system) rule of *reductio ad absurdum* (RAA) also uses DNE essentially, hence, it is not intuitionistically acceptable.

RAA

$$
\begin{array}{l|l}
n & \neg\phi \\
\hline
\vdots & \vdots \\
m & \bot \\
\vdots & \vdots \\
\end{array}
$$
$$
\begin{array}{ll}
s & \phi
\end{array}
$$

Natural Deduction

The derivable (in the given system) rule of *ex falso quodlibet sequitur* (meaning 'from a contradiction everything follows') is also characteristic of classical logic, as it distinguishes it from other logics, crucially, paraconsistent logics.
(Ex falso is also intuitionistically valid.)

Ex falso

$$\begin{array}{l|ll} \vdots & \vdots & \\ n & \bot & \\ \vdots & \vdots & \\ m & \psi & \neg\text{E}, n, m \end{array}$$

Exercise: Prove that $\bot \vdash \psi$ for arbitrary ψ in classical ND.

Natural Deduction

Universal Quantifier

$$
\begin{array}{l|ll}
n & \phi(a) & \\
\vdots & \vdots & \\
m & \forall x \phi(x) & \forall\text{I},\ n
\end{array}
\qquad\qquad
\begin{array}{l|ll}
n & \forall x \phi(x) & \\
\vdots & \vdots & \\
m & \phi(a) & \forall\text{E},\ n
\end{array}
$$

Where in the rule $\forall I$, a cannot appear in any premise or assumption active at line m.

Natural Deduction

Existential Quantifier

n	$\phi(a)$	
$\vdots$	$\vdots$	
m	$\exists x \phi(x)$	$\exists$I, n

n	$\exists x \phi(x)$		
$\vdots$	$\vdots$		
m		$\phi(a)$	
$\vdots$		$\vdots$	
s		χ	
r	χ		$\exists$E, n, m–s

Where in the rule $\exists E$, a cannot appear earlier in the derivation, and it cannot appear in the formula χ derived at line r.

Axiomatic system

Axiomatic System for Classical Propositional Logic:

- Rule: Modus Ponens[3]
- Axioms: The result of *substituting* wffs for ϕ, ψ, and χ in any of the following schemas is an axiom.
  1. $\phi \to (\psi \to \phi)$
  2. $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$
  3. $(\neg\psi \to \neg\phi) \to ((\neg\psi \to \phi) \to \psi)$

Exercise: Give an axiomatic derivation of $p \to p$ and one in ND.

[3] Modus Ponens is the same as the ND Elimination rule for implication.

$\vdash p \rightarrow p$

1. $p \rightarrow ((q \rightarrow p) \rightarrow p)$ $\quad$ Ax_1
2. $(p \rightarrow ((q \rightarrow p) \rightarrow p)) \rightarrow ((p \rightarrow (q \rightarrow p)) \rightarrow (p \rightarrow p))$ $\quad$ Ax_2
3. $(p \rightarrow (q \rightarrow p)) \rightarrow (p \rightarrow p)$ $\quad$ MP 1, 2
4. $(p \rightarrow (q \rightarrow p)$ $\quad$ Ax_1
5. $p \rightarrow p$ $\quad$ MP 3, 4

$$
\begin{array}{l|l l}
1 & p & \\
\cline{2-2}
2 & p & \\
\multicolumn{1}{l}{3} & \multicolumn{1}{l}{p \rightarrow p} & \Rightarrow\mathrm{I},\ 1,\ 2
\end{array}
$$

Consistency

Consistency is an important proof-theoretic notion.
We define its negation:

Definition

A set of formulas Γ is inconsistent iff $\Gamma \vdash \bot$.

This means that there is a set $\Gamma' \subseteq \Gamma$ such that for some formula ϕ, both ϕ and its negation $\neg\phi$ are derivable from Γ' in a finite number of steps.

Semantics of Classical Logic

- Truth-conditional semantics: the meaning of a wff is determined by the conditions under which it is true.
- Bivalence: wffs are either *true* (1) or *false* (0) and not both.
- Compositionality: the truth-value of complex wffs is a function of the truth-values of their atomic components and the semantics of the logical operators.

Propositional Logic

Definition

The valuation function V is defined as the function that assigns to each wff one of 1 or 0 in accordance with the following clauses:

- $V(\phi) = 1$ or 0 and not both, for ϕ atomic
- $V(\neg\phi) = 1$ iff $V(\phi) = 0$
- $V(\phi \wedge \psi) = 1$ iff $V(\phi) = 1$ and $V(\psi) = 1$
- $V(\phi \vee \psi) = 1$ iff either $V(\phi) = 1$ or $V(\psi) = 1$
- $V(\phi \rightarrow \psi) = 1$ iff either $V(\phi) = 0$ or $V(\psi) = 1$

Some terminology

Definition

A wff is called a *tautology* or *logical truth* iff it is true under all valuations.

Definition

A wff is called a *contradiction* iff it is false under all valuations.

Definition

A wff is called *satisfiable* iff there is at least one valuation that makes it true.

Definition

A set of wffs Γ is called *satisfiable* iff there is at least one valuation that makes all $\gamma \in \Gamma$ true.

Truth-functional Completeness

ϕ	ψ	$*_1$	$*_2$	$*_3$	$*_4$	$*_5$	$*_6$	$*_7$	$*_8$	$*_9$	$*_{10}$	$*_{11}$	$*_{12}$	$*_{13}$	$*_{14}$	$*_{15}$	$*_{16}$
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
1	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
0	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

Truth-functional Completeness

De Morgan laws:

$$\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$$
$$\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$$

Definition

Two wffs are *logically equivalent* iff they obtain the same truth-value under every valuation.

The following are also truth-functionally complete sets of logical operators: $\{\neg, \wedge\}$, $\{\neg, \vee\}$, $\{\neg, \rightarrow\}$.

First-order Logic

A model $\mathcal{M}$ consists of a domain $\mathcal{D}$, that is, a non-empty set, and an interpretation function $\mathcal{I}$: $\mathcal{M} = \{\mathcal{D}, \mathcal{I}\}$. The interpretation $\mathcal{I}$ is a total function that fixes the meaning of the non-logical expressions of the language, that is, constants and predicates. The assignment g is a total function that assigns objects to the variables of the language. Assignments are necessary in order to fix the meaning of formulas that contain free variables. There are in principle more than one assignments possible relative to a particular model $\mathcal{M}$.

The interpretation function $\mathcal{I}$ is subject to the following constraints:

Definition

- $\mathcal{I}(t) \in \mathcal{D}$, if t is a constant
- $\mathcal{I}(R)$ is a set of ordered n-tuples from the domain $\mathcal{D}$, for R an n-ary relation

First-order Logic

Before giving the valuation function, we define the auxiliary notion of *denotation* $[t]_{\mathcal{M},g}$ of a term t as follows:

Definition

- $[t]_{\mathcal{M},g} = \mathcal{I}(t)$ if t is a constant
- $[t]_{\mathcal{M},g} = g(t)$ if t is a variable

First-order Logic

Valuation function $V_{\mathcal{M},g}$:

Definition

For any wffs ϕ, ψ, n-ary predicate R and terms t_1, ... , t_n:

- $V_{\mathcal{M},g}(R(t_1, ..., t_n)) = 1$ iff $< [t_1]_{\mathcal{M},g}, ..., [t_n]_{\mathcal{M},g} > \in \mathcal{I}(R)$
- $V_{\mathcal{M},g}(\neg\phi) = 1$ iff $V_{\mathcal{M},g}(\phi) = 0$
- $V_{\mathcal{M},g}(\phi \wedge \psi) = 1$ iff $V_{\mathcal{M},g}(\phi) = 1$ and $V_{\mathcal{M},g}(\psi) = 1$
- $V_{\mathcal{M},g}(\phi \vee \psi) = 1$ iff either $V_{\mathcal{M},g}(\phi) = 1$ or $V_{\mathcal{M},g}(\psi) = 1$
- $V_{\mathcal{M},g}(\phi \rightarrow \psi) = 1$ iff either $V_{\mathcal{M},g}(\phi) = 0$ or $V_{\mathcal{M},g}(\psi) = 1$
- $V_{\mathcal{M},g}(\forall x \phi(x)) = 1$ iff for all assignments $g, V_{\mathcal{M},g}(\phi(x)) = 1$
- $V_{\mathcal{M},g}(\exists x \phi(x)) = 1$ iff for some assignment g, $V_{\mathcal{M},g}(\phi(x)) = 1$

We add a clause for the special predicate '$=$' expressing identity:

- $V_{\mathcal{M},g}(t_1 = t_2) = 1$ *iff* $[t_1]_{\mathcal{M},g}$ *and* $[t_2]_{\mathcal{M},g}$ *are the same*

First-order Logic

The notions of *tautology*, *contradiction* and *satisfiability* are now defined relative to all possible models $\mathcal{M}$ and assignments g.

Definition

A set of formulas Γ is *satisfiable* iff there is model $\mathcal{M}$ and assignment g such that for all $\gamma \in \Gamma$, $V_{\mathcal{M},g}(\gamma) = 1$.

Note that the semantics of classical logic is extensional.
Compare with the semantics for knowledge or belief with is the subject matter of epistemic logic (next lecture).

Classical Entailment

The main idea behind entailment is *truth-preservation*:

Definition

An argument $< \Gamma, \phi >$ is semantically valid iff whenever all premises $\gamma \in \Gamma$ are true the conclusion ϕ is also true.

For propositional logic, this means that for any assignment of truth-values to the atomic components of the formulas such that the valuation function decides the premises to be true, the conclusion is also decided to be true. For predicate logic it means that:

Definition

$\Gamma \vDash \phi$ iff for every model $\mathcal{M}$ and assignment g, if $V_{\mathcal{M},g}(\gamma) = 1$ for all $\gamma \in \Gamma$, then $V_{\mathcal{M},g}(\phi) = 1$.

In the special case that $\vDash \phi$, that is, Γ is the empty set, ϕ is a *logical truth*.

Important properties

Theorem

For any $\gamma \in \Gamma$, $\Gamma \vDash \gamma$.

The following property is called *transitivity*:

Theorem

If $\Gamma \vDash \delta$ *and* $\delta \vDash \phi$, *then* $\Gamma \vDash \phi$.

The following property is called *monotonicity*:

Theorem

If $\Gamma \subseteq \Delta$ *and* $\Gamma \vDash \phi$, *then* $\Delta \vDash \phi$.

Important properties

It is a corollary of monotonicity that:

Theorem

If ϕ is a tautology, then $\Gamma \vDash \phi$ for any Γ.

Notice that $\Gamma \vDash \phi$ iff the set $\{\Gamma, \neg\phi\}$ is not satisfiable, which also means that:

Theorem

If Γ is not satisfiable, then $\Gamma \vDash \phi$ for any ϕ.

This last one can be seen as the semantic counterpart of the ex falso rule. The proof is a straightforward application of the definition of entailment.
Exercise: Prove monotonicity for first order classical entailment.

Propositional Logic is *decidable* but..

Definition

A logic is *decidable* if there exists an effective mechanical procedure that decides for arbitrary formulas whether they are logical truths or not.

- This is also studied as the satisfiability (SAT) decision problem.
- For propositional logic such a procedure does exist,
- hence, the logic is *decidable*. But:
- It is an open question today whether there exists an algorithm to terminate this decision process in polynomial time.
- In computational complexity theory, this is the famous open question of whether $P = NP$.

First order logic is *semi-decidable*

- A similar effective mechanical procedure does not exist for first order logic.
- Completeness for first-order logic (Gödel 1929).
- There are effective procedures (proof systems) for proving a formula if it is a logical truth (i.e. finding out in finite time).
- But if it is not, we may never find out, since we'd have to survey an infinite number of (infinite) models.
- So first-order logic is *semi-decidable*: logical truths can be effectively enumerated, not so for non logical truths.
- The question of whether first-order logic is decidable was posed by Hilbert in 1928 as his famous Entscheidungsproblem. It was answered by Church in 1936 and Turing in 1937.

Relating Syntax and Semantics: Derivability and Entailment

Soundness guarantees that all provable formulas are logical truths.

Completeness guarantees that all logical truths are provable.

Classical logic is both sound and complete:

Theorem

$$\vdash \phi \textit{ iff } \vDash \phi$$

Relating Syntax and Semantics: Consistency and Satisfiability

Theorem

A set of formulas Γ *is satisfiable iff it is consistent.*

Proof hint: Soundness is used to prove the left to right direction, completeness for the converse.
Exercise: Complete the proofs.

Compactness

We can now prove another fundamental meta-theoretic result about first order logic, so-called *compactness*:

Theorem

A set of formulas Γ is satisfiable iff every finite Γ' such that $\Gamma' \subseteq \Gamma$ is satisfiable.

Proof

Left to right direction. Take model $\mathcal{M}$ and assignment g such that they satisfy Γ. Then $\mathcal{M}$, g also satisfy every Γ' such that $\Gamma' \subseteq \Gamma$.

Right to left direction. Assume that Γ is not satisfiable. It follows that Γ is inconsistent, that is, there is finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \bot$. It follows that Γ' is not satisfiable. $\square$

Expressive limitations of first order logic

The following theorem is an interesting corollary of compactness.

Theorem

Finiteness cannot be expressed in first order logic.

Proof

Assume there exists a first order formula ϕ such that it is true only in finite models. Take the infinite set $\{\phi, \psi_1, \psi_2, ...\}$, where ψ_1 expresses 'there exists at least one element', ψ_2 expresses 'there exist at least two elements' etc. This set violates compactness. Hence, there is no such formula ϕ. $\square$

Peano Arithmetic (PA)

Language of Peano Arithmetic:

1. Constant 0.
2. 1-place function symbol S. ($S(x)$ for 'the successor of x'.)
3. 2-place function symbols $+$ and $\times$. (We write '$(x + y)$' and '$(x \times y)$' instead of $+(x, y)$ and $\times(x, y)$ respectively.)

Peano Arithmetic (PA)

Axioms of Peano Arithmetic:

1. $\forall x(Sx \neq 0)$
2. $\forall x(Sx \neq x)$
3. $\forall x \forall y(Sx = Sy \rightarrow x = y)$
4. $\forall x((x + 0) = x)$
5. $\forall x \forall y((x + Sy) = S(x + y))$
6. $\forall x((x \times 0) = 0)$
7. $\forall x \forall y((x \times Sy) = (x \times y) + x)$

8. The induction axiom:
   $(\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(S(x))) \rightarrow \forall x \phi(x)$

Existence of non-standard models of PA

Compactness can also be used to prove about the first order theory of Peano arithmetic (PA) that:

Theorem

There exist non-standard models of PA with infinite numbers.

Proof

Take constant c and the infinite set of sentences $\{c \geq n_1, c \geq n_2, \ldots\}$ for all natural numbers n. Every finite subset of this set has a model, therefore, by compactness, the set itself has a model. It follows that c must be larger than any natural number, hence, an infinite number. □

(in recommended order)

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thank you

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